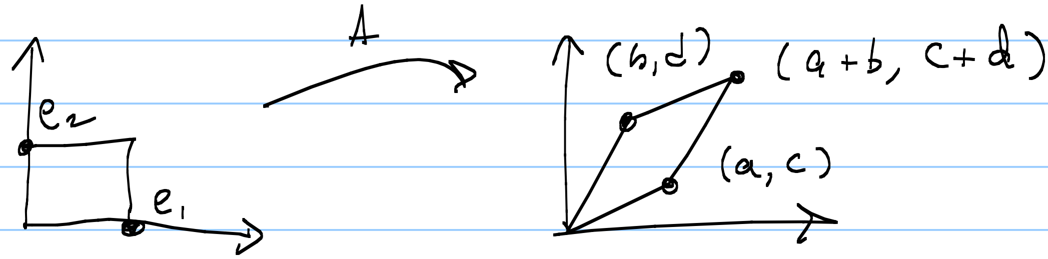
# Jacobian vs. Dilation

Note Title 5/7/2010

The determinant is the area of the parallepided:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A e_1 = \begin{bmatrix} a \\ c \end{bmatrix}, \quad A e_2 = \begin{bmatrix} b \\ d \end{bmatrix}$$



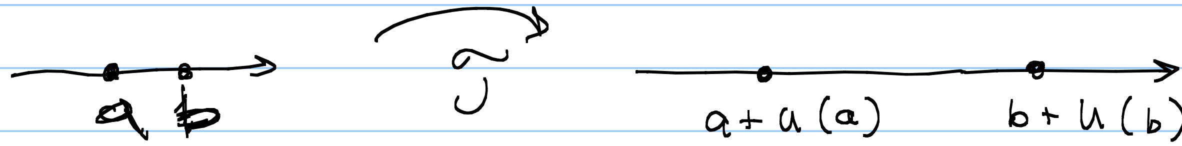

The area of the parallepiped is given as: $ad - bc$, which is $\det A$.

The vector field typically operates on "displacements" rather than transforms. For example:

$$a \longrightarrow a + u(a)$$

where $u(a) = (u_x(a), u_y(a), u_z(a))$

The voxel at $a$ increases or decreases in size according to how the vector field varies. For example consider two voxels at locations $a$ and $b$:




For purposes of computing dilation or expansion, we can use location $a$ as a reference point.

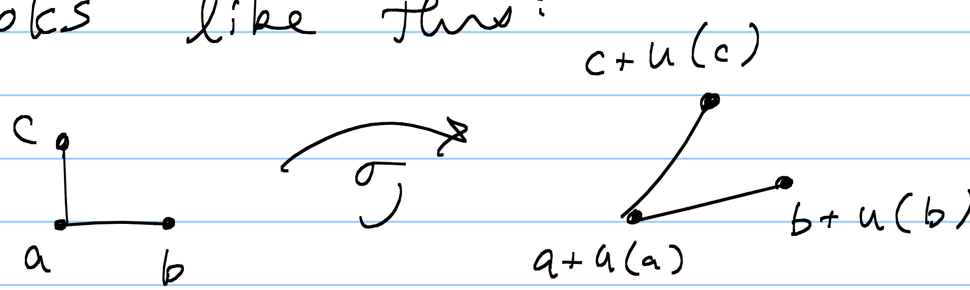
$$T(x) = a + u(a) + A(x - a)$$

where:

$$A = \left[ 1 + \frac{(u(b) - u(a))}{b - a} \right]$$

$$= \left[ 1 + \partial u / \partial x \right]$$

In 2-D, this expansion matrix looks like this:



$$A = I + \begin{bmatrix} \dfrac{u_x(b) - u_x(a)}{b-a} & \dfrac{u_x(c) - u_x(a)}{c-a} \\[2ex] \dfrac{u_y(b) - u_y(a)}{b-a} & \dfrac{u_y(c) - u_y(a)}{c-a} \end{bmatrix}$$

$$= I + \begin{bmatrix} \partial u_x / \partial x & \partial u_x / \partial y \\ \partial u_y / \partial x & \partial u_y / \partial y \end{bmatrix}$$

Just to verify the above definition we plug in and solve at location $b$:

$$\begin{aligned} J(b) &= a + u(a) + A \begin{bmatrix} b-a \\ 0 \end{bmatrix} \\ &= a + u(a) + (b-a) + \begin{bmatrix} u_x(b) - u_x(a) \\ u_y(b) - u_y(a) \end{bmatrix} \\ &= a + (b-a) + u(a) + (u(b) - u(a)) \\ &= b + u(b). \end{aligned}$$

The dilation is computed as

$$\det A$$

Now, the Jacobian ($\det A$) is given as

$$\det A = (1+u_{xx})(1+u_{yy}) - u_{xy}u_{yx}$$

$$= 1 + u_{xx} + u_{yy} + u_{xx}u_{yy} - u_{xy}u_{yx}$$

This can be simplified to something called the "dilation", which ignores the cross-terms

$$\text{dilation} \triangleq u_{xx} + u_{yy}$$

In 3-D, the Jacobian is given as:

$$\begin{aligned}\det A &= (1+u_{xx})(1+u_{yy})(1+u_{zz}) \\ &\quad + u_{xy}u_{yz}u_{zx} + u_{xz}u_{yx}u_{zy} \\ &\quad - (1+u_{xx})u_{yz}u_{zy} - u_{xy}u_{yx}(1+u_{zz}) \\ &\quad - u_{xz}(1+u_{yy})u_{zx}\end{aligned}$$