MI theory V3

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2:39 PM

For standard probabilities we have

$$S = \sum_{i,k} p(i,k|\mu) \ln \frac{p(i,k|\mu)}{p_T(i|\mu)\, p_R(k)}$$

However the above has p's normalized s.t. $\sum p = 1$. We prefer to deal with un-normalized p's, because it allows us to compute everything in one pass.

Let

$$P(i,k|\mu) = N\, p(i,k|\mu)$$

Then:

$$S = \sum_{i,k} \frac{1}{N} P(i,k|\mu) \ln\left(\frac{\frac{1}{N} P(i,k|\mu)}{\frac{1}{N} P_T(i|\mu)\ \frac{1}{N} P_R(k)} \right)$$

$$= \frac{1}{N} \sum P(i,k|\mu) \left(\ln N + \ln \frac{P(i,k|\mu)}{P_T(i|\mu)\, P_R(k)} \right)$$

$p = p(i,k|\mu)$

$t = p_T(i|\mu) = \sum_k p(i,k|\mu)$

$r = p_R(k)$

What is $\underline{\partial S}$?

$\partial p_{i_0 k_0}$

$$\frac{\partial S}{\partial p_{i_0k_0}} = \frac{\partial}{\partial p_{i_0k_0}} \sum_k p_{i_0k} \ln \frac{p_{i_0k}}{t_{i_0} r_k}$$

☆ when $k = k_0$:

$$\frac{\partial}{\partial p_{i_0k_0}}\left(p_{i_0k_0}\ln\frac{p_{i_0k_0}}{t_{i_0}r_{k_0}}\right) = \ln\frac{p_{i_0k_0}}{t_{i_0}r_{k_0}} + p_{i_0k_0}\frac{\partial}{\partial p_{i_0k_0}}\ln\frac{p_{i_0k_0}}{t_{i_0}r_{k_0}}$$

$$= \ln\frac{p}{tr} + p\frac{tr}{p}\frac{\partial}{\partial p}\frac{p}{tr}$$

$$= \ln\frac{p}{tr} + tr\left(\frac{tr - p\frac{\partial}{\partial p}(tr)}{(tr)^2}\right)$$

$$= \ln\frac{p}{tr} + \frac{1}{tr}\left(tr - pr\right)$$

$$= \ln\frac{p}{tr} + \left(\frac{t-p}{t}\right)$$

☆ when $k \neq k_0$ (Notation below is bad...)

$$\frac{\partial}{\partial p_{i_0k_0}}\left(p_{i_0k}\ln\frac{p_{i_0k}}{t_{i_0}r_k}\right) = p_{i_0k}\frac{\partial}{\partial p_0}\ln\frac{p_{i_0k}}{t_{i_0}r_k}$$

$$= p\frac{tr}{p}\frac{\partial}{\partial p_0}\frac{p_{i_0k}}{t_{i_0}r_k}$$

$$= pt\frac{\partial}{\partial p_0}\frac{1}{t}$$

$$= pt\left(-\frac{1}{t^2}\right)\cancelto{1}{\frac{\partial}{\partial p_0}t}$$

$$= -p/t$$

Putting it together

$$\frac{\partial S}{\partial p_{i_0k_0}} = \ln\frac{p_{i_0k_0}}{t_{i_0}r_{k_0}} + \left(\frac{t_{i_0}-p_{k_0}}{t_{i_0}}\right) - \sum_{k\neq k_0}\frac{p_{i_0k}}{t_{i_0}}$$

$$\ldots r_{i_0 k_0} \quad \ldots \quad t_{i_0} r_{k_0} \quad \left(\frac{\quad}{t_{i_0}} \right) \quad \sum_{k \neq k_0} \frac{\quad}{t_{i_0}}$$

$$= \ln \frac{P}{tr} + 1 - \sum_k \frac{P_{i_0 k}}{t_{i_0}}$$

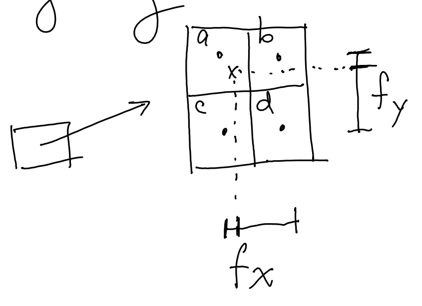
$$= \ln \frac{P}{tr} \quad \triangle$$

RMK: This does not match the result in Maes. But it is experimentally verified.

For comparison, Maes has:

$$\ln \frac{P}{tr} - I$$

What is $\frac{\partial p}{\partial x}$? It depends upon the update rule. For volume averaging we have:



$$\left.\begin{aligned} \frac{\partial a}{\partial x} &= \frac{\partial c}{\partial x} = -P_x \\ \frac{\partial b}{\partial x} &= \frac{\partial d}{\partial x} = +P_x \end{aligned}\right\} \text{ these are valid for small } dx.$$
